Supplemental Information

Toward a precise theory of imprecise representations

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Optimization problem: solution

The endogenous efficient-coding optimization problem is

$$\min_{n,I_1} L_{f,m,p}[nI_1] + \lambda n \text{ subject to } \int \sqrt{I_1(x)}\mathrm{d}x \le \sqrt{K}. \tag{1}$$

where

$$L_{f,m,p}[nI_1] = \frac{1}{p}\int \frac{f(x)^m}{(nI_1(x))^{p/2}}\mathrm{d}x. \tag{2}$$

Solving first for $I_1(x)$, we obtain

$$I_1(x) = K\frac{f(x)^\gamma}{Z^2_{f,\gamma}}, \tag{3}$$

where

$$\gamma = \frac{2m}{p+1} \tag{4}$$

and

$$Z_{f,\gamma} = \int f(x)^{\gamma/2}\mathrm{d}x. \tag{5}$$

Substituting $I_1(x)$ in the optimization problem above, we obtain the problem for n:

$$\min_n \frac{Z^{p+1}_{f,\gamma}}{p\,K^{p/2}}\frac{1}{n^{p/2}} + \lambda n. \tag{6}$$

Solving for n, we obtain

$$n = \left(\frac{Z^{p+1}_{f,\gamma}}{2\lambda K^{p/2}}\right)^{\frac{2}{p+2}}. \tag{7}$$

Thus the total Fisher information $I(x) = nI_1(x)$ is

$$I(x) = \frac{f(x)^{\gamma_{task}}}{C_{f,m,p}}, \tag{8}$$

where

$$C_{f,m,p} = \left(\frac{2\lambda Z_{f,\gamma}}{K}\right)^{\frac{2}{p+2}}. \tag{9}$$

Bias and imprecision

Given the approximation to the bias,

$$bias(x) = \frac{1}{I(x)}\left[\frac{f'(x)}{f(x)} - \frac{I'(x)}{I(x)}\right], \tag{10}$$

the bias with the optimal encoding (Eq. 8) is approximately

$$bias_\gamma(x) = (1-\gamma)C_{f,m,p}\frac{f'(x)}{f(x)^{\gamma+1}}. \tag{11}$$

Thus if $\gamma < 1$ the bias Equivalently,

$$bias_\gamma(x) = C_{f,m,p}\,\frac{\gamma-1}{\gamma}\,\frac{\mathrm{d}}{\mathrm{d}x}f(x)^{-\gamma}. \tag{12}$$

With a task characterized by m', p', and $\gamma = \frac{2m'}{p'+1}$, we have

$$\sigma^2_{\gamma'}(x) = \frac{1}{I(x)} = C_{f,m',p'}f(x)^{-\gamma'}. \tag{13}$$

Therefore,

$$bias_\gamma(x) = \mathcal{C}\frac{\mathrm{d}}{\mathrm{d}x}\,\sigma_{\gamma'}(x)^{\,2\gamma/\gamma'}, \tag{14}$$

where

$$\mathcal{C} = \frac{\gamma-1}{\gamma}\frac{C_{f,m,p}}{C^{\gamma/\gamma'}_{f,m',p'}}. \tag{15}$$